

# SOME REMARKS ON T-COPULAS

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## Abstract

We examine three methods of constructing correlated Student- $t$  random variables. Our motivation arises from simulations that utilise heavy-tailed distributions for the purposes of stress testing and economic capital calculations for financial institutions. We make several observations regarding the suitability of the three methods for this purpose.

**Keywords:** Student- $t$  distribution, correlation, copula.

## 1 Introduction

The use of heavy-tailed distributions for the purposes of stress testing and economic capital calculations has gained attention recently in an attempt to capture exposure to extreme events.

Among the various distributions available, the Student- $t$  distribution has gained popularity in these calculations for several reasons (as opposed to, say,  $\alpha$ -stable distributions). The first is that for three or more degrees of freedom it possesses a finite variance, and so can be calibrated to the variance of observable data. The second is that  $t$ -variables are relatively easy and fast to generate for simulations.

However, one very desirable property that should be exhibited by any calculation of economic capital is the ability to capture concentrated risks. Put simply, asset movements - particularly large movements - should be correlated. Thus, it is necessary to generate correlated  $t$ -variables. A recent paper on this topic is [SL] - we refer the reader to this paper for the necessary background on  $t$ -copulas, and the references contained therein.

In this paper we examine three  $t$ -copulas in this context, in particular their properties regarding correlation and tail correlation.

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## 2 T-Copulas

Let  $X, Y \sim N(0, 1)$  with correlation  $\rho(X, Y) = \rho$ .

Typically, correlated Student- $t$  distributions with  $n$  degrees of freedom,  $U$  and  $V$ , can be formed via the transformations:

$$U = X \sqrt{\frac{n}{C}}, \quad V = Y \sqrt{\frac{n}{C}} \quad (2.1)$$

where  $C$  is sampled from a chi-squared distribution with  $n$  degrees of freedom<sup>1</sup>

An alternative formulation is given by:

$$U = X \sqrt{\frac{n}{C_1}}, \quad V = Y \sqrt{\frac{n}{C_2}} \quad (2.2)$$

where  $C_1$  and  $C_2$  are independently sampled from a chi-squared distribution with  $n$  degrees of freedom. This formulation is suggested to be more desirable in [SL] as it gives rise to a product structure of the density function when  $\rho = 0$ .

However, we will show that this has a major impact on the correlation, and in particular the resulting bivariate distribution<sup>2</sup> and tail correlation.

Another naïve method of constructing correlated  $t$ -variables,  $U$  and  $V$ , (assumed to have the same degrees of freedom) is the following: take uncorrelated  $t$ -variables,  $U$  and  $W$ , then put

$$V = \rho U + \sqrt{1 - \rho^2} W \quad (2.3)$$

However,  $V$  will not have a  $t$ -distribution as the sum of two  $t$ -variables is not a  $t$ -variable. Note that for three degrees of freedom or more, the  $t$ -variable sums lie within the domain of attraction of the Normal distribution. However, since we are only performing one sum, the tail of the distribution is still a power law of order  $n$ . Despite this, the resulting distribution does possess some useful properties.

(2.1), (2.2), and (2.3) define the three  $t$ -copulas that we will examine. We refer to these  $t$ -copulas as being generated by the *Same*  $\chi^2$ , *Independent*  $\chi^2$ , and *Correlated- $t$* , respectively.

## 3 Independent $\chi^2$

We will firstly examine the case of the Independent  $\chi^2$   $t$ -variables. We now show that this construction has a major impact on the correlation as follows:

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<sup>1</sup>Formulations for Student- $t$  distributions with different degrees of freedom can be found in [SL].

<sup>2</sup>We shall restrict our analysis in this paper to the bivariate case only.

Let  $A = \sqrt{\frac{n}{C_1}}$  and  $B = \sqrt{\frac{n}{C_2}}$ . We have,

$$\rho(U, V) = \frac{\mathbb{E}(UV) - \mathbb{E}(U)\mathbb{E}(V)}{\sqrt{\text{Var}(U)\text{Var}(V)}} \quad (3.1)$$

$$= \frac{\mathbb{E}(XAYB) - \mathbb{E}(XA)\mathbb{E}(YB)}{\sqrt{\text{Var}(XA)\text{Var}(YB)}} \quad (3.2)$$

$$= \frac{\mathbb{E}(XY)\mathbb{E}(A)\mathbb{E}(B) - \mathbb{E}(X)\mathbb{E}(A)\mathbb{E}(Y)\mathbb{E}(B)}{\sqrt{\text{Var}(X)\text{Var}(Y)\text{Var}(A)\text{Var}(B)}} \quad (\text{by independence})$$

$$= \frac{\mathbb{E}(XY)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} \frac{\mathbb{E}(A)\mathbb{E}(B)}{\sqrt{\text{Var}(A)\text{Var}(B)}} \quad (\text{Since } \mathbb{E}(X) = \mathbb{E}(Y) = 0)$$

$$= \rho \frac{\mathbb{E}(A)\mathbb{E}(B)}{\sqrt{\text{Var}(A)\text{Var}(B)}} \quad (3.3)$$

Assuming that  $A$  and  $B$  have the same distribution, we have

$$\rho(U, V) = \rho \frac{\mathbb{E}(A)\mathbb{E}(B)}{\sqrt{\text{Var}(A)\text{Var}(B)}} \quad (3.4)$$

$$= \rho \frac{\mathbb{E}(A)^2}{\mathbb{E}(A^2)} \quad (3.5)$$

$$< \rho \quad (\text{by Jensen})$$

In fact, the amount by which the correlation is reduced, namely

$$\frac{\mathbb{E}(A)^2}{\mathbb{E}(A^2)} = \frac{\mathbb{E}(n/C_1)^2}{\mathbb{E}((n/C_1)^2)} = \frac{\mathbb{E}(1/C_1)^2}{\mathbb{E}((1/C_1)^2)} \quad (3.6)$$

can be determined explicitly. For the case where  $C_1$  and  $C_2$  have 3 degrees of freedom, this turns out to be  $2/\pi \approx 0.6366$ .

We now determine the amount by which the correlation is reduced by explicitly. We first begin with a calculation of the required moments - we could not find a convenient reference, and record it here for completeness:

**Lemma 1.** *The  $n$ th moment of the Inverse-Chi Distribution with  $\nu$  degrees of freedom is given by:*

$$\mathbb{E}(Y^n) = \frac{\Gamma((\nu - n)/2)}{\Gamma(\nu/2)} 2^{n/2} \quad (3.7)$$

**Proof:** Let  $f(x; \alpha, \beta)$  denote the Gamma distribution, given by

$$f(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-x/\beta) \quad (3.8)$$

If  $\alpha = \nu/2$  and  $\beta = 2$ , then this is the chi-squared distribution with  $\nu$  degrees of freedom.

We wish to make the transformation  $Y = 1/\sqrt{X}$ . Since this is a monotonic function, we use the transformation formula:

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| \quad (3.9)$$

Thus,

$$f_Y(y; \alpha, \beta) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| \quad (3.10)$$

$$= f_X\left(\frac{1}{y^2}\right) \left| \frac{-2}{y^3} \right| \quad (3.11)$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} (y^{-2})^{\alpha-1} \exp(-(y^{-2})/\beta) \frac{2}{y^3} \quad (3.12)$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} y^{-2\alpha-1} \exp(-1/(\beta y^2)) \quad (3.13)$$

$$(3.14)$$

We now derive the formula for the  $n$ -th moment of  $Y$ . Firstly, note that

$$\int_0^\infty y^{-2\alpha-1} \exp(-1/(\beta y^2)) dy = \frac{\Gamma(\alpha)}{\beta^\alpha} \quad (3.15)$$

and we have:

$$\mathbb{E}(Y^n) = \int_0^\infty y^n \frac{\beta^\alpha}{\Gamma(\alpha)} y^{-2\alpha-1} \exp(-1/(\beta y^2)) \quad (3.16)$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty y^{-2(\alpha-n/2)-1} \exp(-1/(\beta y^2)) \quad (3.17)$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha - n/2)}{\beta^{\alpha-n/2}} \quad (3.18)$$

$$= \frac{\Gamma(\alpha - n/2)}{\Gamma(\alpha)} \beta^{n/2} \quad (3.19)$$

**Remark:** Note that the moments for this distribution will not be defined when  $\alpha - n/2$  is a negative integer.

Consider now the case when  $\alpha = \nu/2$  and  $\beta = 2$ :

$$\mathbb{E}(Y^n) = \frac{\Gamma((\nu - n)/2)}{\Gamma(\nu/2)} 2^{n/2} \quad (3.20)$$

as required.  $\square$

**Proposition 1.** *The factor by which the correlation is reduced by is given by*

$$\frac{[\mathbb{E}(Y)]^2}{\mathbb{E}(Y^2)} = \left( \frac{\Gamma((\nu - 1)/2)}{\Gamma(\nu/2)} \right)^2 \times \left( \frac{\nu - 2}{2} \right) \quad (3.21)$$

Furthermore, we have for  $\nu = 3$ ,

$$\frac{[\mathbb{E}(Y)]^2}{\mathbb{E}(Y^2)} = \frac{2}{\pi} \approx 0.6366 \quad (3.22)$$

and for large  $\nu$  we have

$$\frac{[\mathbb{E}(Y)]^2}{\mathbb{E}(Y^2)} = \frac{\nu - 2}{\nu - 1} \rightarrow 1 \text{ as } \nu \rightarrow \infty \quad (3.23)$$

**Proof:** Let us first consider the case  $\nu = 3$ :

$$\mathbb{E}(Y^n) = \frac{\Gamma((3-n)/2)}{\Gamma(3/2)} 2^{n/2} \quad (3.24)$$

Hence,

$$\mathbb{E}(Y) = \frac{\Gamma(1)}{\Gamma(3/2)} 2^{1/2} = \frac{1}{\sqrt{\pi}/2} 2^{1/2} = \frac{2\sqrt{2}}{\sqrt{\pi}} \quad (3.25)$$

and

$$\mathbb{E}(Y^2) = \frac{\Gamma(1/2)}{\Gamma(3/2)} 2 = \frac{\sqrt{\pi}}{\sqrt{\pi}/2} 2 = 4 \quad (3.26)$$

Thus we have,

$$\frac{[\mathbb{E}(Y)]^2}{\mathbb{E}(Y^2)} = \left( \frac{2\sqrt{2}}{\sqrt{\pi}} \right)^2 \times \left( \frac{1}{4} \right) = \frac{2}{\pi} \approx 0.6366 \quad (3.27)$$

For general degrees of freedom we have

$$\frac{[\mathbb{E}(Y)]^2}{\mathbb{E}(Y^2)} = \left( \frac{\Gamma((\nu-1)/2)}{\Gamma(\nu/2)} 2^{1/2} \right)^2 \times \left( \frac{\Gamma(\nu/2)}{\Gamma((\nu-2)/2)} 2^{-2/2} \right) \quad (3.28)$$

$$= \Gamma^2((\nu-1)/2) \times \left( \Gamma((\nu-2)/2) \Gamma(\nu/2) \right)^{-1} \quad (3.29)$$

$$= \Gamma^2((\nu-1)/2) \times \left( 2/(\nu-2) \times \Gamma^2(\nu/2) \right)^{-1} \quad (3.30)$$

$$= \left( \frac{\Gamma((\nu-1)/2)}{\Gamma(\nu/2)} \right)^2 \times \left( \frac{\nu-2}{2} \right) \quad (3.31)$$

**Remark:** Compare (3.31) with the expression given in [SL], section 4.1, which is very similar, except that they (erroneously) give the square-root of this expression.

Using the properties of Beta functions and Stirling's formula, we have that

$$\frac{\Gamma((\nu-1)/2)}{\Gamma(\nu/2)} = \frac{B(\frac{1}{2}, \frac{\nu-1}{2})}{\sqrt{\pi}} \quad (3.32)$$

$$= \sqrt{\frac{2}{\nu-1}} \quad (3.33)$$

and thus

$$\frac{[\mathbb{E}(Y)]^2}{\mathbb{E}(Y^2)} = \frac{\nu-2}{\nu-1} \quad (3.34)$$

as required.  $\square$

To explicitly compute (3.31) for a given value of  $\nu$ , we need to consider odd and even cases. We table here the following values:

Degrees of Freedom	Reduction of Correlation
3	0.6366
4	0.7854
5	0.8488
6	0.8836
7	0.9054
8	0.9204
9	0.9313
10	0.9396
20	0.9726
50	0.9896
100	0.9949

## 4 Empirical Distributions and Copulas

We now provide empirical results for each of our three t-copulas. We simulated 1,000,000 observations using each of the three methods, and have provided below graph of the pdf's of the distribution and the copulas. We have only considered here the case of the t-distribution with three degrees of freedom, and a base correlation of 0.9. (the graphs presented in this section are based on the first 5,000 observations)

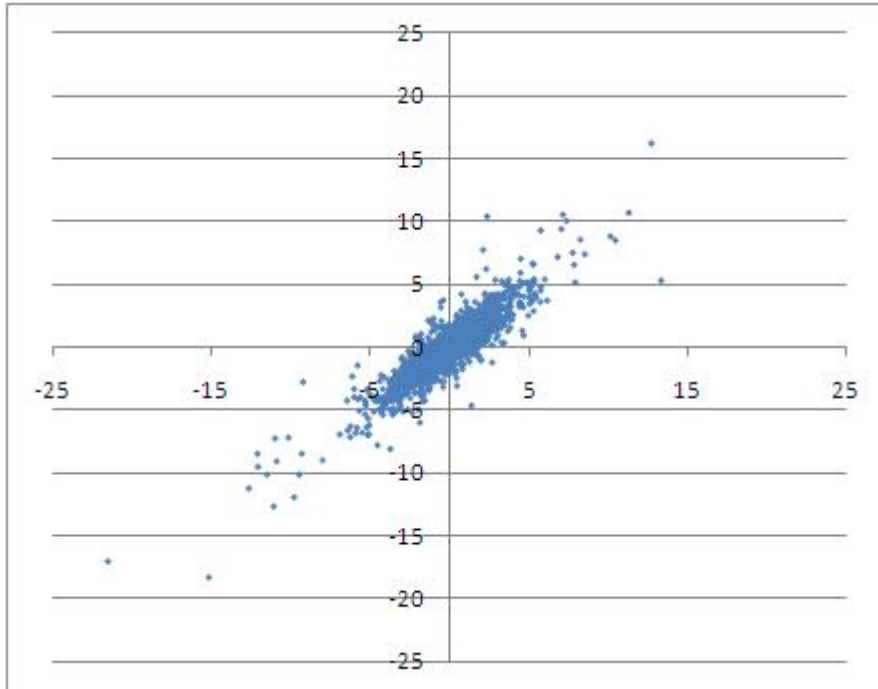


Figure 1: Same  $\chi^2$

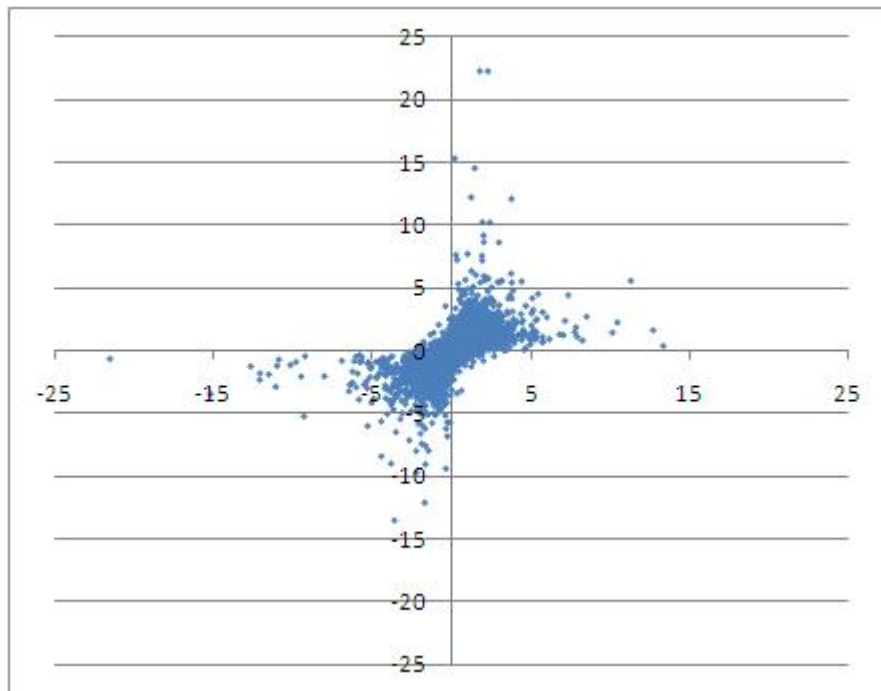


Figure 2: Independent  $\chi^2$

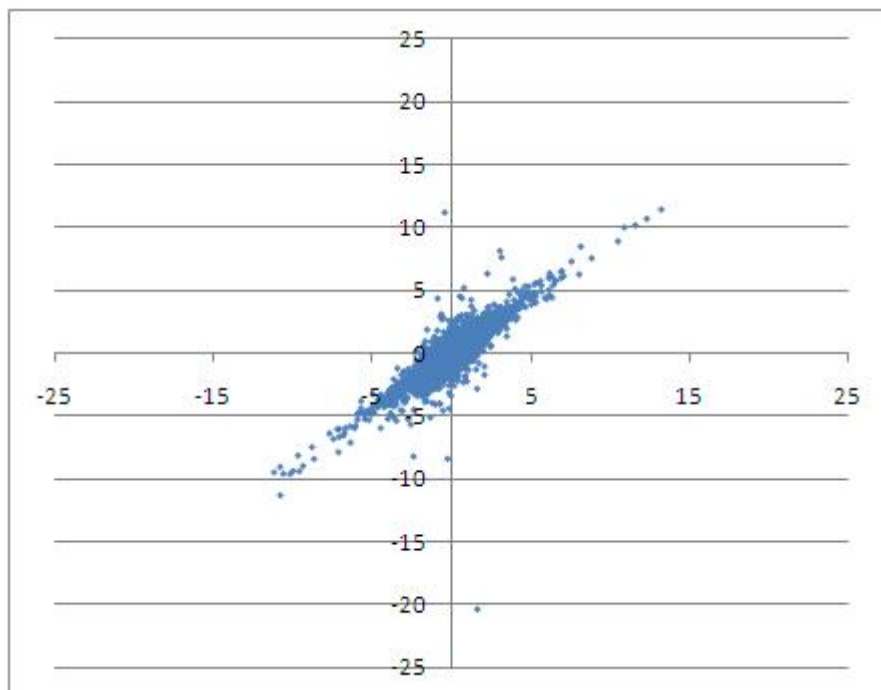


Figure 3: Correlated- $t$

As can be clearly seen, the pdf's of the Same  $\chi^2$  and Correlated- $t$  are elliptical, but the pdf of the Independent  $\chi^2$  is quite splayed out.

We have also constructed graphs of their copulas (below). Despite being slightly more “fatter”, the copula of the Independent  $\chi^2$  appear little different.

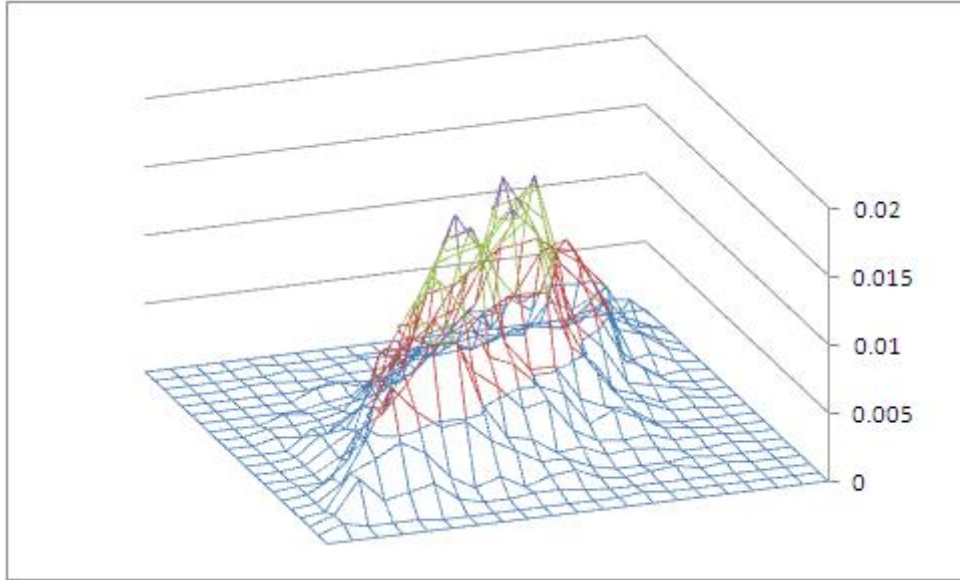


Figure 4: Same  $\chi^2$  Copula Density



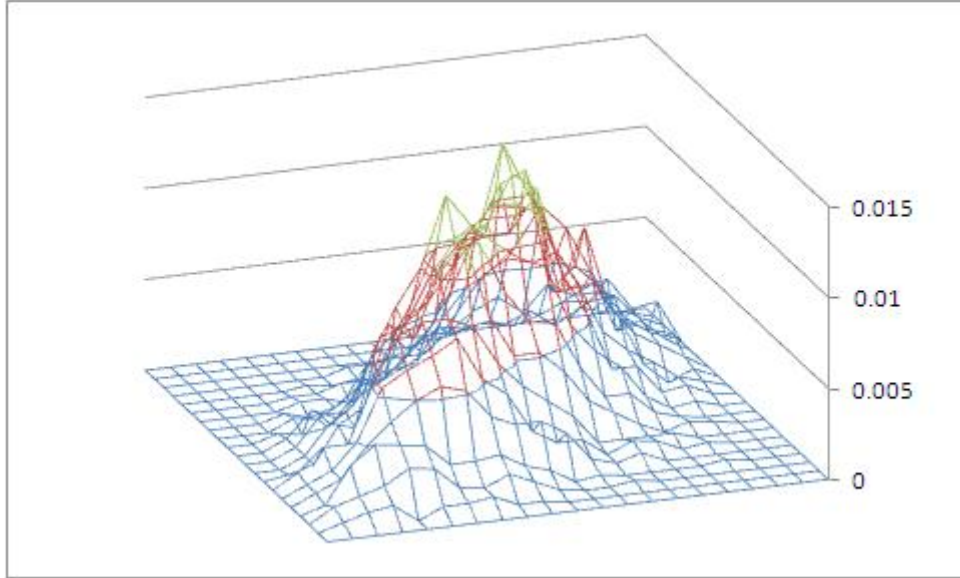


Figure 5: Independent  $\chi^2$  Copula Density

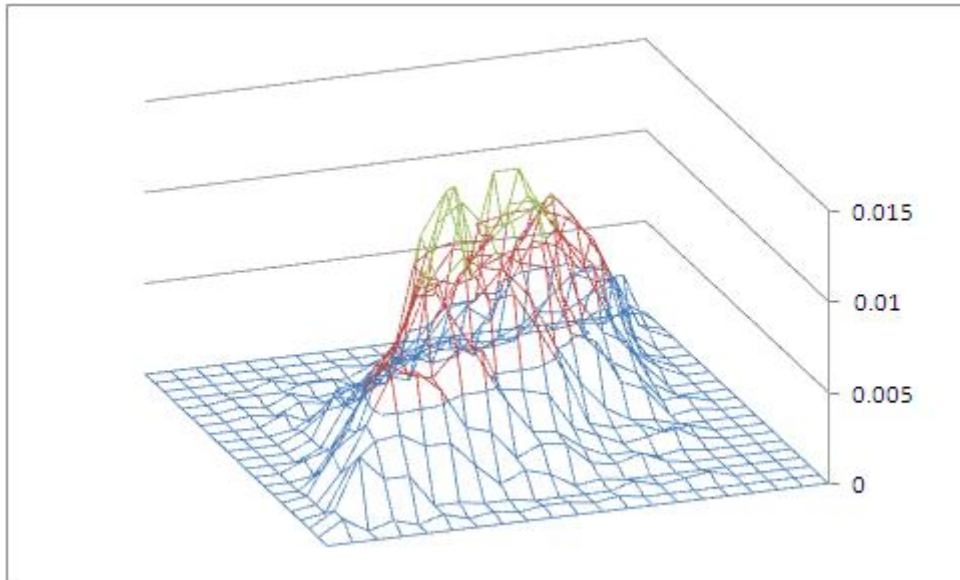


Figure 6: Correlated- $t$  Copula Density

## 5 Tail Correlation

We firstly prove a general result concerning the tail correlation. Recall that the tail correlation is the quantity

$$\text{correl}(U, V | U > \gamma) \quad (5.1)$$

where  $\gamma$  is the number of standard deviations into the tail.

Consider two random variables,  $Z$  and  $X$ , having correlation  $\rho$ , formed by the sum:

$$Z = \rho X + \sqrt{1 - \rho^2} Y \quad (5.2)$$

where  $X$  and  $Y$  are independent random variables.

**Theorem 1.** *The tail correlation of  $X$  and  $Z$  is given by*

$$\text{correl}(X, Z | X > \mu) = \frac{1}{\sqrt{1 + \frac{K'}{V}}} \quad (5.3)$$

where

$$V = \text{Var}[X | X > \mu] \quad \text{and} \quad K' = \text{Var}[Y] / \rho^2 \quad (5.4)$$

Furthermore, suppose the tails of  $X$  and  $Y$  have a power law, then

$$\text{correl}(X, Z | X > \mu) \rightarrow 1 \quad \text{as} \quad \mu \rightarrow \infty \quad (5.5)$$

and  $X$  and  $Y$  are Normal variables, then

$$\text{correl}(X, Z | X > \mu) \rightarrow 0 \quad \text{as} \quad \mu \rightarrow \infty \quad (5.6)$$

The corollary for the  $t$ -distribution follows from our proof of this theorem:

**Corollary 1.** *Let  $X$  and  $Y$  be independent  $t$ -variables with  $\nu$  degrees of freedom. Then the tail variance of  $X$  is given by*

$$V = \text{Var}[X | X > \mu] = \mu^2 \frac{\nu}{(\nu - 1)^2 (\nu - 2)} \quad (5.7)$$

and therefore

$$\text{correl}(X, Z | X > \mu) = \frac{1}{\sqrt{1 + \frac{K'(\nu - 1)^2 (\nu - 2)}{\mu^2 \nu}}} \quad (5.8)$$

**Proof:** Consider the conditional correlation, given  $X > \mu$ . Then we have

$$\begin{aligned} \mathbb{E}[XZ | X > \mu] &= \rho \mathbb{E}[X^2 | X > \mu] + \sqrt{1 - \rho^2} \mathbb{E}[XY | X > \mu] \\ &= \rho \mathbb{E}[X^2 | X > \mu] \end{aligned}$$

$$\mathbb{E}[X | X > \mu] \mathbb{E}[Z | X > \mu] = \rho \mathbb{E}[X | X > \mu]^2 \quad (5.9)$$

$$\text{Var}[X | X > \mu] = \mathbb{E}[X^2 | X > \mu] - \mathbb{E}[X | X > \mu]^2 = V \quad (\text{say}) \quad (5.10)$$

$$\begin{aligned}
Var[Z|X > \mu] &= \rho^2 Var[X|X > \mu] + (1 - \rho^2) Var[Y|X > \mu] \\
&= \rho^2 V + (1 - \rho^2) Var[Y|X > \mu] \\
&= \rho^2 V + K
\end{aligned} \tag{say}$$

and

$$correl(XZ) = \frac{\rho V}{\sqrt{V} \sqrt{\rho^2 V + K}} = \frac{1}{\sqrt{1 + \frac{K'}{V}}} \tag{5.11}$$

so the behaviour of the tail correlation depends on the tail variance  $V$ . We firstly examine power law tails, then the Normal distribution.

### Power law tails:

Let the tail of a distribution (density) be  $f(x) = Cx^{-n}$ . The conditional distribution is:

$$F(x|X > \mu) = \frac{Pr(\mu < X < x)}{Pr(X > \mu)} = \frac{\mu^{-n+1} - x^{-n+1}}{\mu^{-n+1}} \tag{5.12}$$

and the conditional density is:

$$f(x|X > \mu) = \frac{(n-1)x^{-n}}{\mu^{-n+1}} \tag{5.13}$$

Thus, the conditional variance (that is,  $V$ ) is given by

$$\begin{aligned}
Var[X|X > \mu] &= \mathbb{E}[X^2|X > \mu] - \mathbb{E}[X|X > \mu]^2 \\
&= \int_{\mu}^{\infty} \frac{(n-1)x^{-n+2}}{\mu^{-n+1}} dx - \left( \int_{\mu}^{\infty} \frac{(n-1)x^{-n+1}}{\mu^{-n+1}} dx \right)^2 \\
&= \frac{(n-1)\mu^{-n+3}}{(n-3)\mu^{-n+1}} - \left( \frac{(n-1)\mu^{-n+2}}{(n-2)\mu^{-n+1}} \right)^2 \\
&= \frac{(n-1)}{(n-3)} \mu^2 - \frac{(n-1)^2}{(n-2)^2} \mu^2 \\
&= \frac{n-1}{(n-2)^2(n-3)} \mu^2 \rightarrow \infty \quad \text{as } \mu \rightarrow \infty
\end{aligned}$$

and thus

$$correl(XZ|X > \mu) = \frac{1}{\sqrt{1 + \frac{K'}{V}}} \rightarrow 1 \quad \text{as } \mu \rightarrow \infty \tag{5.14}$$

### Normal Distribution:

The conditional distribution for the Normal distribution is:

$$F(x|X > \mu) = \frac{Pr(\mu < X < x)}{Pr(X > \mu)} = \frac{N(x) - N(\mu)}{1 - N(\mu)} \tag{5.15}$$

and the conditional density is:

$$f(x|X > \mu) = \frac{n(x)}{1 - N(\mu)} \quad (5.16)$$

Now

$$\int_{\mu}^{\infty} x^2 n(x) dx = \mu n(\mu) + 1 - N(\mu) \quad (5.17)$$

and

$$\int_{\mu}^{\infty} x n(x) dx = n(\mu) \quad (5.18)$$

Thus, the conditional variance (that is,  $V$ ) is given by

$$\begin{aligned} Var[X|X > \mu] &= \mathbb{E}[X^2|X > \mu] - \mathbb{E}[X|X > \mu]^2 \\ &= \frac{1}{1 - N(\mu)} \left[ \int_{\mu}^{\infty} x^2 n(x) dx - \frac{1}{1 - N(\mu)} \left( \int_{\mu}^{\infty} x n(x) dx \right)^2 \right] \\ &= \frac{\mu n(\mu) + 1 - N(\mu)}{1 - N(\mu)} - \frac{n(\mu)^2}{[1 - N(\mu)]^2} \\ &= 1 + \frac{\mu n(\mu)}{1 - N(\mu)} - \frac{n(\mu)^2}{[1 - N(\mu)]^2} \end{aligned}$$

It can be shown using l'hôpital's rule that this expression tends to zero as  $\mu$  tends to infinity. Thus we have,

$$correl(XZ) = \frac{1}{\sqrt{1 + \frac{K'}{V}}} \rightarrow 0 \quad \text{as} \quad \mu \rightarrow 0 \quad (5.19)$$

which concludes the proof.  $\square$

## 6 Tail Correlations: Empirical Results

We now examine numerical calculations of the tail correlations for each of our three t-copulas. That is, we estimate the quantity

$$correl(U, V | U > \gamma) \quad (6.1)$$

where  $\gamma$  is the number of standard deviations into the tail. We record the tail correlation for each of our three t-copulas - using the same base 0.9 correlation and three degrees of freedom - for the given standard deviation below:

Tail St. Dev.	Tail Correlation		
	Same $\chi^2$	Different $\chi^2$	Correlated-t
$\gamma$			
2	0.8273	0.0430	0.9314
3	0.8255	0.0124	0.9562
4	0.8210	-0.0046	0.9723
5	0.8168	-0.0046	0.9800
6	0.8113	-0.0011	0.9839
7	0.8074	-0.0021	0.9880
8	0.8049	-0.0148	0.9899
9	0.8036	-0.0199	0.9912
10	0.8044	-0.0128	0.9927
11	0.8033	-0.0024	0.9934
12	0.8003	-0.0031	0.9942
13	0.8015	-0.0094	0.9941
14	0.8053	-0.0132	0.9942
15	0.8023	-0.0120	0.9938
16	0.8027	-0.0132	0.9942
17	0.8016	-0.0206	0.9938
18	0.7991	-0.0247	0.9945
19	0.7983	-0.0390	0.9944
20	0.7927	-0.0338	0.9949

As can be seen, the tail correlation for the t-distribution using the Same  $\chi^2$  remains high, and decays slightly into the tail to approximately 0.8. Unsurprisingly for the t-distribution using Different  $\chi^2$ , the tail correlation is quite low, and even becomes slightly negative into the tail. This is a feature of the tail, and can be seen in Figure 7 below, which is a graph of all observations where the first variable. As predicted by Theorem 1, the tail correlation of the Correlated-t *increases* into the tail.

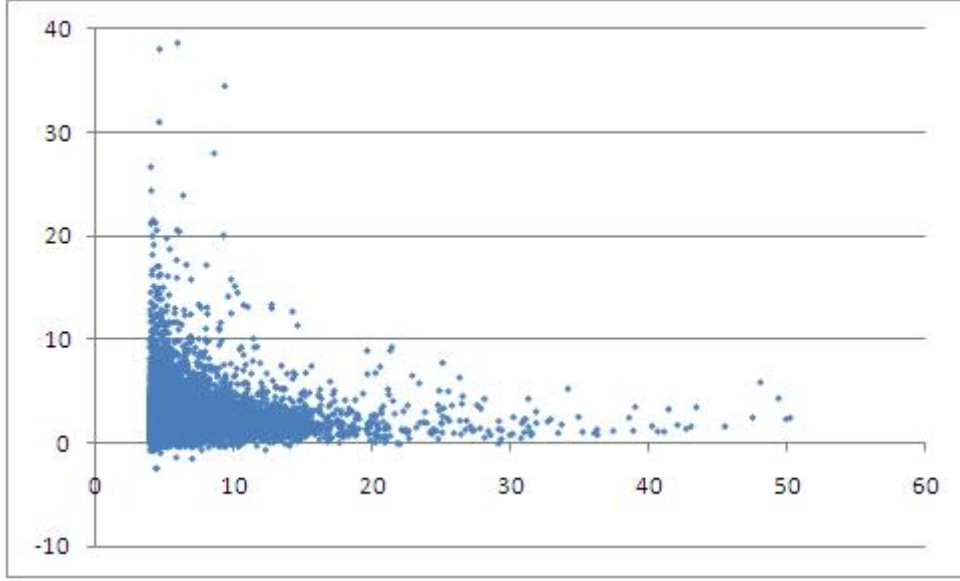


Figure 7: Independent  $\chi^2$  Tail

We also give the number of tail observations (out of 1,000,000) for each of our three t-copulas, again using the same base 0.9 correlation and three degrees of freedom. That is, the quantity

$$\#(U, V \mid U > \gamma, V > \gamma) \quad (6.2)$$

Tail St. Dev.	Tail Observations		
	Same $\chi^2$	Different $\chi^2$	Correlated-t
2	49925	24825	53046
3	19918	5558	21880
4	9534	1524	10472
5	5199	451	5706
6	3099	165	3421
7	2039	71	2214
8	1417	37	1530
9	998	22	1088
10	726	9	800
11	556	5	606
12	439	3	476
13	334	0	383
14	258	0	305
15	208	0	255
16	166	0	204
17	143	0	171
18	118	0	147
19	97	0	127
20	88	0	113

## 7 Summary and Conclusions

We have examined the t-copulas for the purposes of stress testing and economic capital calculations. It appears that using correlated t-variables generated by using different  $\chi^2$  is not appropriate as the correlation is affected (and essentially destroyed) by the construction. Whilst the Correlated-t is not a true t-distribution with the desired degrees of freedom, the distribution is still heavy tailed, and has the desired properties regarding correlation and in particular tail correlation.

## References

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